Electric Flux & Flux Density (1)

→ Faraday's experiment, inner sphere given a charge Q, outer spheres (two hemi-spheres) uncharged.

→ $-Q$ coulombs charge induced on outer spheres

→ More charge on inner spheres, more flux lines and more charge induced

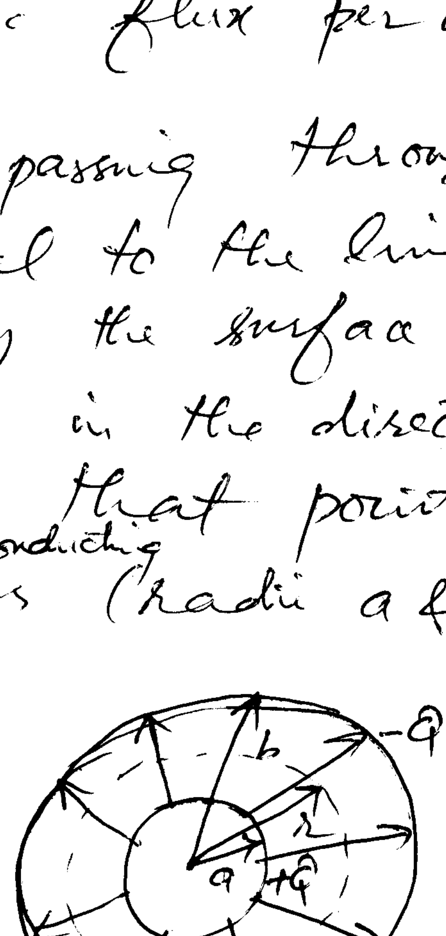
→ Hence $\psi = Q$

→ D electric flux density
defined as electric flux per unit area

→ No. of flux lines passing through a surface normal to the lines of flux, divided by the surface area

→ direction of $\vec{D}$ is in the direction of flux lines at that point.

→ Considering two metallic conducting spheres (radii a & b) charges $+Q$ and $-Q$.



$$\vec{D}\Big|_{r=a} = \frac{Q}{4\pi a^2}\,\hat{a}_r$$

$$\vec{D}\Big|_{r=b} = \frac{Q}{4\pi b^2}\,\hat{a}_r$$

$$\vec{D}\Big|_{r} = \frac{Q}{4\pi r^2}\,\hat{a}_r \quad , \quad a \le r \le b$$

Comparing it with,

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{a}_r$$

$$\Rightarrow \boxed{\vec{D} = \epsilon_0 \vec{E}} \rightarrow \text{defining eq. of } \vec{D} \text{ in free space.}$$

Generally,

$$\vec{E} = \int_{vol} \frac{\rho_v dv}{4\pi\epsilon_0 R^2}\hat{a}_R$$

$$\rightarrow \vec{D} = \int_{vol} \frac{\rho_v dv}{4\pi R^2}\hat{a}_R$$

→ $\vec{D}$ is associated with the concept of divergence

→ $\vec{D}$ fields are relatively simpler than $\vec{E}$ fields.

__Example__: Find $\vec{D}$ in the region about a uniform line of charge of 8 nC/m lying along z-axis in free space.

__Sol__

$$\vec{E} = \frac{\rho_L}{2\pi\epsilon_0 \rho}\hat{a}_\rho = \frac{8\times10^{-9}}{2\pi\epsilon_0 \rho}\hat{a}_\rho = \frac{143.8}{\rho}\hat{a}_\rho \text{ V/m}$$

$$\vec{D} = \frac{\rho_L}{2\pi\rho}\hat{a}_\rho = \frac{1.273\times10^{-9}}{\rho} \text{ C/m}^2$$

At $\rho = 3$ m,

$$\vec{E} = 47.9\,\hat{a}_\rho \text{ V/m}$$

$$\vec{D} = 0.424\,\hat{a}_\rho \text{ nC/m}^2$$

→ The total flux leaving a 5-m length of the line charge is equal to the total charge on that length, or $\Psi = 40$ nC

Gauss's Law

→ The electric flux passing through any closed surface is equal to the charge enclosed by that surface.

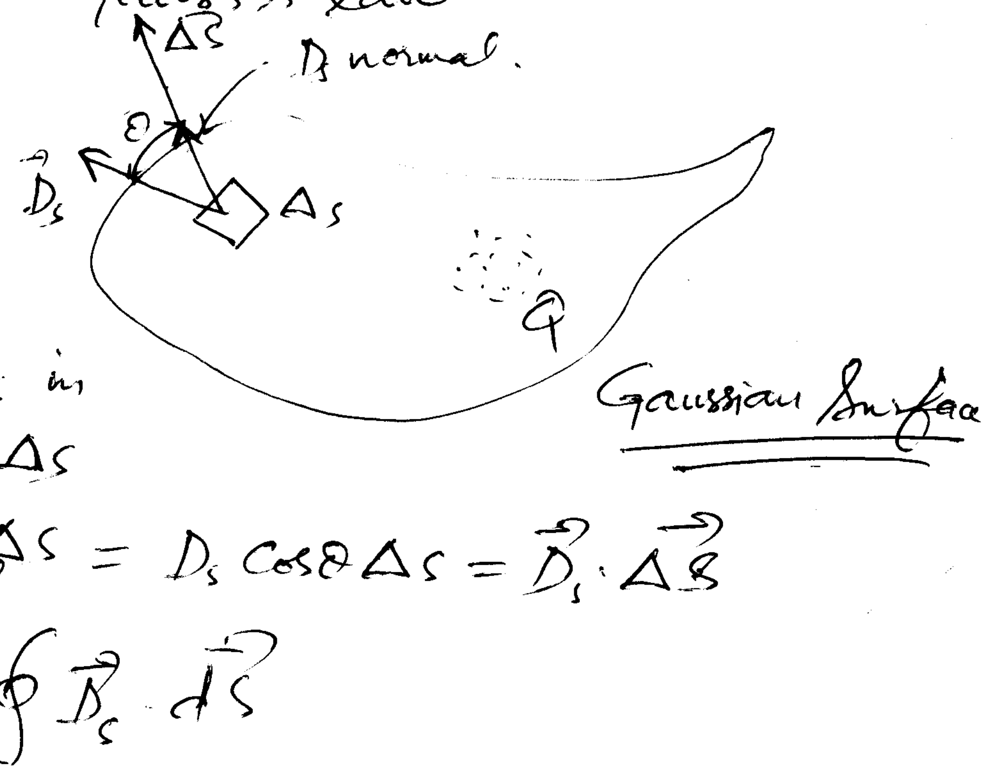
→ Charge may be distributed over a surface (of any shape) or it may be concentrated, but the total flux will be the same.

→ Flux density may be different depending upon the shape of charged surface.

→ Derivation of Gauss's law

→ Incremental area of surface ΔS

* direction outward normal
* Component of D_s in the direction of ΔS



$$\Delta\psi = D_{s_{normal}}\Delta S = D_s \cos\theta \Delta S = \vec{D}_s \cdot \Delta\vec{S}$$

$$\psi = \int d\psi = \oint_s \vec{D}_s \cdot d\vec{S}$$

$\vec{ds}$ → $dx\,dy$, $\rho\, d\phi\, d\rho$, $r^2 \sin\theta\, d\theta\, d\phi$ (double integral)

→ Integration over a closed surface.

$$\psi = \oint_s \vec{D}_s \cdot d\vec{S} = Q = \text{charged enclosed}$$

→ $Q = \sum Q_n$, $\int \rho_L dL$, $\int_s \rho_s dS$, $\int_{Vol} \rho_v dV$

General Form

$$\oint_s \vec{D}_s \cdot \vec{ds} = \int_{vol} \rho_v \, dV$$

→ A <u>point charge</u> placed at the centre (origin) of a spherical coordinate system

★ Assuming a Gaussian surface of radius 'a' (a spherical surface).

★ At the surface.

$$\vec{D} = \frac{Q}{4\pi a^2} \hat{a}_r$$

$$\vec{ds} = r^2 \sin\theta \, d\theta \, d\phi = a^2 \sin\theta \, d\theta \, d\phi \, \hat{a}_r$$

$$\Rightarrow \vec{D} \cdot \vec{ds} = \frac{Q}{4\pi a^2} \cdot a^2 \sin\theta \, d\theta \, d\phi \, \hat{a}_r \cdot \hat{a}_r$$

$$\vec{D} \cdot \vec{ds} = \frac{Q}{4\pi} \sin\theta \, d\theta \, d\phi$$

$$\Rightarrow \oint_s \vec{D} \cdot \vec{ds} = \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} \frac{Q}{4\pi} \sin\theta \, d\theta \, d\phi$$

$$= \int_0^{2\pi} \frac{Q}{4\pi} (-\cos\theta) \Big|_0^{\pi} d\phi = \int_0^{2\pi} \frac{Q}{2\pi} d\phi$$

$$= Q$$

Which verifies our ~~result~~ law as the charge enclosed by the surface is Q coulombs.

Application Of Gauss's Law
Symmetrical Charge Distributions

→ Problem is to find D_s using Gauss's law

$$Q = \oint_S \vec{D}_s \cdot d\vec{s}$$ (Unknown quantity inside the integral)

→ Strategy to hit

- Choose a closed surface satisfying two conditions

① $\vec{D}_s$ is everywhere either normal or tangential to the closed surface, so that $\vec{D}_s \cdot d\vec{s}$ becomes either $D_s\, dS$ or zero respectively.

② On that portion of the closed surface for which $\vec{D}_s \cdot d\vec{s}$ is not zero, D_s = constant.

→ $\oint \vec{D}_s \cdot d\vec{s} = D_s \oint_S ds$

→ Knowledge of symmetry is important to choose such a surface.

→ Example of a point charge when sperical surface is assumed.

→ Uniform linear charge distribution lying along the z-axis ($-\infty$ to $+\infty$)

- only the radial component of $\vec{D}$ is present.

i.e $\vec{D} = D_\rho \hat{a}_\rho$

$D_\rho = f(\rho)$

- Choice of closed surface is easy, a cylinderical surface to which D_ρ is everywhere normal.
- A right circular cylinder of radius ρ

$$Q = \oint_{cyl} \vec{D}_s \cdot d\vec{S} = D_s \int_{sides} dS + 0 \int_{top} dS + 0 \int_{bottom} dS$$

$$Q = D_s \int_{z=0}^{L} \int_{\phi=0}^{2\pi} \rho \, d\phi \, dz = D_s (2\pi\rho) L$$

$$\Rightarrow \quad D_s = D_\rho = \frac{Q}{2\pi\rho L} = \frac{\rho_L L}{2\pi\rho L} = \frac{\rho_L}{2\pi\rho}$$

$$\Rightarrow \quad D_s = \frac{\rho_L}{2\pi\rho}$$

$$\& \quad E_\rho = \frac{\rho_L}{2\pi\epsilon_0 \rho}$$

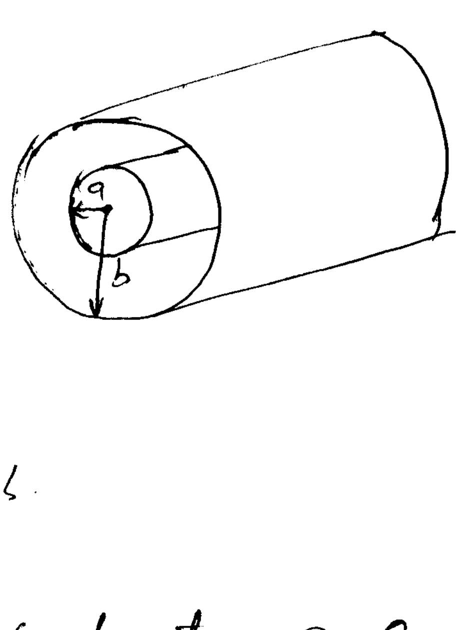
→ Coaxial Cable

→ Two coaxial ~~circular~~ cylinderical conductors of radii a & b.

→ ρ_s is the charge density on the outer surface of inner cylinder

→ Again only radial component is present D_ρ.

→ Gaussian surface is chosen as a right circular cylinder of radius ρ where $a < \rho < b$.



→ $Q = (D_s)(2\pi\rho L)$

Total charge on the inner conductor of length L is

$$Q = \int_{z=0}^{L} \int_{\phi=0}^{2\pi} \rho_s \, a \, d\phi \, dz = 2\pi a L \rho_s$$

$$\Rightarrow \quad (D_s)(\cancel{2\pi}\rho \cancel{L}) = \cancel{2\pi} a \cancel{L} \rho_s$$

$$D = \frac{a\rho_s}{\rho} \quad \text{or} \quad \vec{D} = \frac{a\rho_s}{\rho}\hat{a}_\rho \qquad (a<\rho<b)$$

→ In terms of linear charge density ρ_L

$$Q = D_s 2\pi\rho L \Rightarrow \frac{Q}{L} = D_s 2\pi\rho \Rightarrow \boxed{\rho_L = D_s 2\pi\rho}$$

Also $Q = 2\pi a L\rho_s \Rightarrow \frac{Q}{L} = 2\pi a\rho_s \Rightarrow \boxed{\rho_L = 2\pi a\rho_s}$

Comparing $\rho_s = \frac{\rho_L}{2\pi a}$

$$\Rightarrow \vec{D} = \frac{\cancel{a}}{\rho}\cdot\frac{\rho_L}{2\pi\cancel{a}}\hat{a}_\rho = \frac{\rho_L}{2\pi\rho}\hat{a}_\rho$$

$\boxed{\vec{D} = \frac{\rho_L}{2\pi\rho}\hat{a}_\rho}$ → Solution identical to that of infinite line of charge.

→ Total charge on the inner surface of outer ~~sphere~~ cylindrical conductor

$$Q_{outer\ cyl} = -(2\pi a L\rho_{s,inner\ cyl})$$

Also $2\pi b L\rho_{s_{outer}} = -2\pi a L\rho_{s_{inner}}$

$\Rightarrow \boxed{\rho_{s_{outer}} = -\frac{a}{b}\rho_{s_{inner}}}$ → This implies that charge density on the outer ~~surface~~ cylinder will decrease as it's radius increases and vice versa.

→ Choosing the Gaussian Surface (Cylinder) of $\rho > b$, the total charge enclosed is zero, hence

$$Q = 0 = D_s 2\pi\rho L \Rightarrow \boxed{D_s = 0}$$

→ Similarly for $\rho < a$, $D_s = 0$ (no charge enclosed)

→ Coaxial cable or capacitor has no external field (outer cylinder acts as shield)

No field within the centre conductor.

→ Result is applicable also to a finite length of coaxial cable open at both ends provided that L is many times greater than b.

Example 3.2

$$L = 50\,cm = 0.5\,m$$
$$a = 1\,mm = 1\times10^{-3}\,m$$
$$b = 4\,mm = 4\times10^{-3}\,m$$
$$Q = 30\times10^{-9}\,C$$
$$\rho_s, \vec{E}, \vec{D} = ?$$

$$\rho_{s,inner} = \frac{Q_{inner}}{2\pi a L} = 9.55\,\mu C/m^2$$

$$\rho_{s,outer} = \frac{Q_{outer}}{2\pi b L} = -2.39\,\mu C/m^2$$

$$D_\rho = \frac{a\rho_s}{\rho} = \frac{10^{-3}(9.55\times10^{-6})}{\rho} = \frac{9.55}{\rho}\,n\,C/m^2$$

$$E_\rho = \frac{D_\rho}{\epsilon_0} = \frac{1079}{\rho}\,V/m$$

Application of Gauss's Law:

Differential Volume Element

(Unsymmetrical problem)

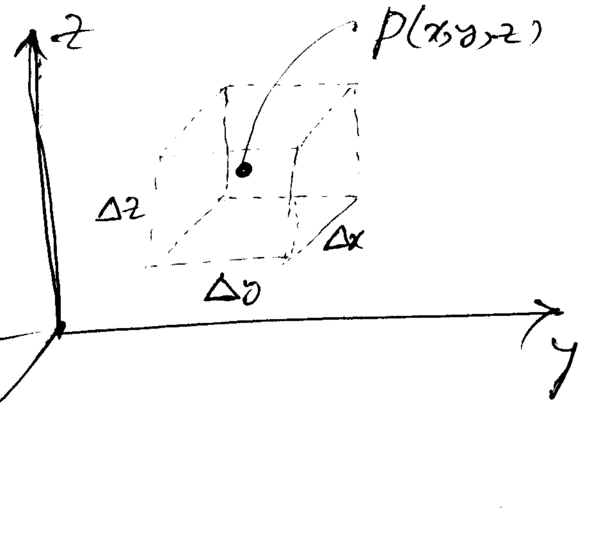
→ Choosing such a very small closed surface for which $\vec{D}$ is almost constant

→ Small change in $\vec{D}$ may be represented by using the 1st two terms of Tailor Series expansion for $\vec{D}$. (9)

→ $\Delta v \to 0$

→ We will not find $\vec{D}$ here but the way $\vec{D}$ varies in this small region.

(Maxwells 1st eq.)



$$\vec{D}_0 = D_{x0}\hat{a}_x + D_{y0}\hat{a}_y + D_{z0}\hat{a}_z$$

→ Our small Gaussian surface is the rectangular box of lengths $\Delta x, \Delta y, \Delta z$

$$\oint \vec{D}\cdot d\vec{S} = Q$$

$$\oint \vec{D}\cdot d\vec{S} = \int_{front} + \int_{back} + \int_{left} + \int_{right} + \int_{top} + \int_{bottom}$$

$$\int_{front} = \vec{D}_{front}\cdot \Delta S_{front} = \vec{D}_{front}\cdot \Delta y \Delta z\, \hat{a}_x$$

$$= (D_{x,front})\Delta y \Delta z$$

Now the front face is at a distance of $\frac{\Delta x}{2}$ from P

$$\Rightarrow D_{x,front} = D_{x0} + \frac{\Delta x}{2} \times \text{Rate of change of } D_x \text{ w.r.t } x$$

$$= D_{x0} + \frac{\Delta x}{2}\cdot\frac{\partial D_x}{\partial x}$$

(D_x in general also varies with y & z)

$$\Rightarrow \int_{front} = \left(D_{x_0} + \frac{\Delta x}{2}\,\frac{\partial D_x}{\partial x}\right)\Delta y \Delta z$$

Similarly

$$\int_{back} = D_{back} \cdot \Delta S_{back}$$

$$= D_{back} \cdot (-\Delta y \Delta z \, a_x)$$

$$= -D_{x,back}\, \Delta y \Delta z$$

$$D_{x,back} = D_{x_0} - \frac{\Delta x}{2} \cdot \frac{\partial D_x}{\partial x}$$

$$\Rightarrow \int_{back} = \left(-D_{x_0} + \frac{\Delta x}{2} \cdot \frac{\partial D_x}{\partial x}\right)\Delta y \Delta z$$

$$\int_{front} + \int_{back} = \frac{\partial D_x}{\partial x}\,\Delta x \Delta y \Delta z$$

$$\int_{right} + \int_{left} = \frac{\partial D_y}{\partial y}\,\Delta x \Delta y \Delta z$$

$$\int_{top} + \int_{bottom} = \frac{\partial D_z}{\partial z}\,\Delta x \Delta y \Delta z$$

$$\Rightarrow \oint_S \vec{D} \cdot d\vec{S} = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right)\Delta x \Delta y \Delta z$$

or

$$Q = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right)\Delta v$$

$\Rightarrow$ Charge enclosed in volume Δv $= \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) \times \Delta v$

Example 3.3 $\vec{D} = e^{-x}\sin y\, \hat{a}_x - e^{-x}\cos y\, \hat{a}_y + 2z\, \hat{a}_z$ C/m^2

$Q = ?$, $\Delta v = 10^{-9}\ m^3$

$\Rightarrow \frac{\partial D_x}{\partial x} = -e^{-x}\sin y$

$\frac{\partial D_y}{\partial y} = e^{-x}\sin y$

$\frac{\partial D_z}{\partial z} = 2$

At the origin

$\frac{\partial D_x}{\partial x} = \frac{\partial D_y}{\partial y} = 0$

$\frac{\partial D_z}{\partial z} = 2$

$\Rightarrow \boxed{Q = 2\Delta v = 2\,nC}$

Divergence

$\rightarrow$ $$\left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right)\Delta v = \oint_s \vec{D}\cdot\vec{ds} = Q$$

or $$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \frac{\oint_s \vec{D}\cdot\vec{ds}}{\Delta v} = \frac{Q}{\Delta v}$$

or As a limit

$$\left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) = \lim_{\Delta v \to 0} \frac{\oint_s \vec{D}\cdot\vec{ds}}{\Delta v} = \lim_{\Delta v \to 0} \frac{Q}{\Delta v}$$

$$\Rightarrow \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) = \rho_v$$

$\rightarrow$ For any vector $\vec{A}$

$$\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) = \lim_{\Delta v \to 0} \frac{\oint_s \vec{A}\cdot\vec{ds}}{\Delta v}$$

or $$\text{div}\, A = \lim_{\Delta v \to 0} \frac{\oint_s \vec{A}\cdot\vec{ds}}{\Delta v}$$

Similarly $$\text{div}\, D = \lim_{\Delta v \to 0} \frac{\oint_s \vec{D}\cdot\vec{ds}}{\Delta v} = \rho_v$$

→ The divergence of $\vec{A}$ is the outflow of flux from a small closed surface per unit volume as the volume shrinks to zero.

→ The divergence of velocity of water in a bathtub is zero b/c water is incompressible and water entering and leaving the closed surface is the same.

→ Air is compressible, divergence from a punctured type is +ve.

→ +ve divergence source

→ -ve divergence sink.

→ zero divergence No source or sink.

→ $$\text{div } \vec{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z}$$

$$\text{div } \vec{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta D_\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial \phi}$$

→ Divergence is an operation performed on a vector but result is a scalar.

→ Divergence merely tells us how much flux is leaving a small volume on a per-unit volume basis; no direction is associated with it.

Maxwell's First Equation
(Electrostatics)

→ $$\mathrm{div}\,\vec{D} = \rho_v$$

it states that the electric flux per unit volume leaving a vanishly small volume unit is exactly equal to the volume charge density there.

→ Point form of Gauss's law

→ Maxwell's 1st eq. is the differential eq. form of Gauss's law

→ Gauss's law is the integral form of Maxwell's 1st equation

Vector Operator ∇ and the divergence Theorem

$$\nabla = \frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z$$

$$\vec{\nabla}\cdot\vec{D} = \left(\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z\right)\cdot\left(D_x\hat{a}_x + D_y\hat{a}_y + D_z\hat{a}_z\right)$$

$$\vec{\nabla}\cdot\vec{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \mathrm{div}\,\vec{D}$$

We can use it only for rectangular coordinates.

Divergence Theorem

$$\oint_s \vec{D}\cdot\vec{ds} = Q$$

$$Q = \int_{vol} \rho_v dv$$

$$\vec{\nabla}\cdot\vec{D} = \rho_v$$

$$\Rightarrow \oint_s \vec{D}\cdot\vec{ds} = Q = \int_{Vol} \rho_v dv = \int_{Vol} \vec{\nabla}\cdot\vec{D}\, dv$$

$$\Rightarrow \boxed{\oint_s \vec{D}\cdot\vec{ds} = \int_{vol} \vec{\nabla}\cdot\vec{D}\, dv} \quad (\text{divergence theorem})$$

"The integral of the normal component of any vector field over a closed surface is equal to the integral of the divergence of this vector field throughout the volume enclosed by the closed surface".

(OR)

"The total flux crossing the closed surface is equal to the integral of the divergence of the flux density throughout the enclosed volume."

Advantage: This theorem relates a triple integration throughout some volume to a double integration over the surface of that volume.

Interpretation of Divergence operation

Definition of divergence:

$$\operatorname{div} \vec{D} = \lim_{\Delta v \to 0} \frac{\oint_S \vec{D} \cdot d\vec{S}}{\Delta v}$$

→ Divergence of a vector field in rectangular coordinates:

$$\operatorname{div} \vec{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

→ Maxwell's 1st eq. applied to Electrostatics and steady magnetic fields:

$$\operatorname{div} \vec{D} = \rho_v \qquad \text{(Point form of Gauss's law)}$$

" Electric flux per unit volume leaving a vanishingly small volume unit is exactly equal to the volume charge density there."

→ Comparison b/w Gauss's law & Maxwell's 1st eq.
Gauss's law relates the flux leaving any closed surface to the charge enclosed, whereas Maxwell's 1st eq. makes an identical statement on a per-unit-volume basis for a vanishingly small volume or at a point.

...tor Operator ∇

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z$$

$\vec{\nabla}\cdot\vec{D}$ = Dot operation and not the dot product.

$$= \left(\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z\right)\cdot\left(D_x\hat{a}_x + D_y\hat{a}_y + D_z\hat{a}_z\right)$$

$$\vec{\nabla}\cdot\vec{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \operatorname{div}\vec{D}$$

$$\Rightarrow \quad \boxed{\operatorname{div}\vec{D} = \vec{\nabla}\cdot\vec{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}}$$

→ ∇ operator has no specific form in other coordinate systems.

→ If $\vec{D}$ is given in cylinderical coordinates then $\vec{\nabla}\cdot\vec{D}$ still indicates the divergence of $\vec{D}$ ($\operatorname{div}\vec{D}$)

→ $$\nabla u = \left(\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z\right)u$$

$$\nabla u = \frac{\partial u}{\partial x}\hat{a}_x + \frac{\partial u}{\partial y}\hat{a}_y + \frac{\partial u}{\partial z}\hat{a}_z$$

Where u is a scalar field.